Permutations

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

The symbol $^{n}P_{r}$ means the number of permutations of r out of n. The symbol "!" represents "factorial" so 5! is 5 factorial. This means to multiply 5 by all positive integers of a smaller value than 5. So 5! = 5×4×3×2×1. For example, if we want to find the number of ways one can arrange 3 letters out of 5.
$^{5}P_{3} = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\times4\times3\times2\times1}{2\times1} = \frac{120}{2} = 60$

Say our 5 letters are ABCDE. Examples of permutations might be:
ABC (1)
ACB (2)
BAC (3)
BDC (4)
etc.

Note how (1), (2) and (3) contain the same letters but in a different order.

Proof

If we were to find the number of ways for arranging all five letters in order, there are 5 letters to choose from for the first position, 4 letters to choose from for the second position, 3 for the third and so on. This would give us 5×4×3×2×1, which is 5!
So $n! = n \times (n-1) \times \ldots \times 2 \times 1$

But say we wanted to find the numbers of ways of arranging three of these five letters. There are 5 letters to choose from for the first position, 4 letters to choose from for the second position, 3 letters for the third position and then we run out of positions. So $^{5}P_{3} = 5 \times 4 \times 3$
This is different from 5! as there is no ×2 but we can write $^{5}P_{3} = \frac{5\times4\times3\times2\times1}{2\times1}$. Obviously, multiplication by 1 gives the same number, so this has no impact on the values. But by having two in the numerator and denominator $^{5}P_{3} = \frac{5\times4\times3\times\cancel{2}}{\cancel{2}}$ the 2s cancel and we are left with $5 \times 4 \times 3$.
So $^{n}P_{r} = n \times (n-1) \times \ldots \times (n-r+2) \times (n-r+1)$
or $^{n}P_{r} = \frac{n\times(n-1)\times\ldots\times2\times1}{(n-r)\times(n-r-1)\times\ldots\times2\times1}$
so $^{n}P_{r} = \frac{n!}{(n-r)!}$

Combinations

$$^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

The symbol $^{n}C_{r}$ means the number of combinations of r out of n. Which is the number of ways n different objects can (irrespective of order) r out of them.

As shown above, ABC and ACB are permutations of each other. But they are not distinct combinations. They are, in fact, the same combination as order is not considered. Different combinations must contain different objects.

Proof

Say we need to find $^{5}C_{3}$
(The number of ways of arranging 3 out of 5) = (The number of ways of choosing 3 out of 5) × (The number of ways of arranging those chosen)
So $^{n}P_{r} = {}^{n}C_{r} \times r!$
So $^{n}C_{r} = \frac{^{n}P_{r}}{r!}$
So $^{n}C_{r} = \frac{n!}{r!(n-r)!}$

Special cases
As established, arranging 5 objects in order is 5! or 120. This can also be represented as $^{5}P_{5} = \frac{5!}{(5-5)!} = \frac{5!}{0!}$ as we already know this must be equal to 5!, it is therefore convenient to define 0! = 1.

See also
- Binomial Expansions
- Probability

References
Turner, L. K. (1976). *Advanced Mathematics*. London: Longman. pp.229-231.